On the Use of Krasovskii's Theorem for Stability Analysis

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In order to establish a finite region of asymptotic stability (RAS) for systems involving chemical reaction in either a single stage or a series of stages, many investigators have used Krasovskii's theorem (*1* to *3*, *5*, *6*). Krasovskii's theorem suggests that for nonlinear stationary systems a satisfactory Liapunov function is a quadratic form in the time derivatives of the state variables. If certain sign definiteness conditions can be met by this Liapunov function and its time derivative, then asymptotic stability of a steady state can be assured. Our purpose here is to point out that certain conclusions drawn from published research might be misconstrued.

The pioneer work in the use of a Krasovskii type of Liapunov function for illustrating a finite RAS was done by Berger and Perlmutter (*1*, *3*), who proposed a hypothetical example of a single stage in which a first-order irreversible exothermic reaction takes place. This example has subsequently been used by others (*4* to *7*). The object in all cases has been to determine a finite RAS by using Krasovskii's theorem, tracking functions, or other methods. None of these authors, however, have indicated that this system is globally stable for all physically meaningful perturbations in the temperature and concentration of the stage, although Paradis and Perlmutter have said that it is possible to demonstrate asymptotic stability for a region of arbitrary size. This conclusion can easily be obtained from the standard plot of heat generated and heat rejected vs. temperature, which shows a single intersection or steady state point. As a consequence, such a globally stable example, although covering many common reactors, does not always provide the most critical test of a method, since such an analysis does not allow for the possibility of multiple steady states.

Using Krasovskii's theorem Berger and Perlmutter (*2*) also investigated the more general case of an arbitrary number of stages in which a single first-order irreversible reaction is taking place. They concluded that for each additional stage after four stages only a single added restriction is needed to ensure the negative definiteness of the time derivative of the Liapunov function. This conclusion is incorrect because it is based upon the expansion of determinants of order greater than 3 by a method which is valid only for determinants of order less than or equal to 3. In fact, their separatrix method requires that two new restrictions must be met to assure asymptotic stability for each added stage independent of the stage number. A recurrence expansion of the system's symmetric five-stripe determinant yields the following two inequality conditions for the i^{th} stage with $j = 2i - 1$ and $j = 2i$:

$$\Delta_j = f_{j,j}\,\Delta_{j-1} - f^2_{j-1,j}\,\Delta_{j-2} - f_{j-1,j-1}\,\Delta_{j-3} + \frac{\Delta_{j-4}}{\tau^4} + 2\,f_{j-1,j}\left\{\sum_{k=4}^{j}(-1)^{j-k+1}\frac{f_{k-3,k-2}\,\Delta_{k-4}}{\tau^{j-k+2}}\right\} > 0$$

with $\Delta_0 = 1$ and $\Delta_l = 0$ for $l < 0$

Finally, it has been stated that the function suggested by the Krasovskii theorem forms a closed surface in the space of the state variables (*1*). This is true only when a globally stable system is encountered. In fact, it can be shown that when multiple steady state points are possible, the closed surface is no longer unique. The presence of a number of closed surfaces, each satisfying the same equation complicates the stability analysis considerably.

In a future publication a method for the determination of an RAS of a nonglobally stable series of stages will be presented. This method, which uses a Krasovskii type of Liapunov function, is based upon the solution of a minimization problem in the n-dimensional space of the state variables.

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